XX. *A new Method of investigating the Sums of Infinite Series.*
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THE fummation of infinite feries is a fubject, not only of
curious fpeculation, but alfo of the greateft importance
in the various branches of mathematics and philofophy; in
confequence of which it has always claimed a very confiderable
fhare of attention from the moft celebrated mathematicians.
I fhall therefore make no apology for offering to the public
the following new and very expeditious method, by which we
may obtain the fums of a great variety of feries, moft of which
have never before been treated of. As the fummation depends
on the fums of the reciprocals of the powers of the natural
numbers, tables of fuch fums are given as far as the 40th
power to twelve places of decimals, by which the fums of
the feries will be found true to ten or eleven places; and if
greater accuracy were required (which is a cafe that can very
rarely happen) it might eafily be obtained by continuing the
tables. The firft and third columns fhew the fums, and the
fecond and fourth the powers correfponding.

TABLE

## TABLE I.

Sum of $\frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} +$ &c. *ad infinitum.*

| Sum | $n =$ | Sum | $n =$ |
|---|---|---|---|
| A =,644934066848 | 2 | W =,000000238450 | 22 |
| B =,202056903159 | 3 | X =,000000119219 | 23 |
| C =,082323233711 | 4 | Y =,000000059608 | 24 |
| D =,036927755107 | 5 | Z =,000000029803 | 25 |
| E =,017343061984 | 6 | A' =,000000014901 | 26 |
| F =,008349277387 | 7 | B' =,000000007450 | 27 |
| G =,004077356198 | 8 | C' =,000000003725 | 28 |
| H =,002008392826 | 9 | D' =,000000001863 | 29 |
| I =,000994575128 | 10 | E' =,000000000931 | 30 |
| K =,000494188604 | 11 | F' =,000000000465 | 31 |
| L =,000246086553 | 12 | G' =,000000000233 | 32 |
| M =,000122713347 | 13 | H' =,000000000116 | 33 |
| N =,000061248135 | 14 | I' =,000000000058 | 34 |
| O =,000030588236 | 15 | K' =,000000000029 | 35 |
| P =,000015282259 | 16 | L' =,000000000015 | 36 |
| Q =,000007637196 | 17 | M' =,000000000007 | 37 |
| R =,000003817292 | 18 | N' =,000000000004 | 38 |
| S =,000001908212 | 19 | O' =,000000000002 | 39 |
| T =,000000953961 | 20 | P' =,000000000001 | 40 |
| V =,000000476932 | 21 | | |

TABLE

## TABLE II.

Sum of $\frac{1}{2^n} - \frac{1}{3^n} + \frac{1}{4^n} - \frac{1}{5^n} + $ &c. *ad infinitum.*

| Sum | $n=$ | Sum | $n=$ |
|---|---|---|---|
| $a$ = ,177532966576 | 2 | $w$ = ,000000238386 | 22 |
| $b$ = ,098457322630 | 3 | $x$ = ,000000119199 | 23 |
| $c$ = ,052967170503 | 4 | $y$ = ,000000059602 | 24 |
| $d$ = ,027880229587 | 5 | $z$ = ,000000029801 | 25 |
| $e$ = ,014448908703 | 6 | $a'$ = ,000000014901 | 26 |
| $f$ = ,007406180072 | 7 | $b'$ = ,000000007450 | 27 |
| $g$ = ,003766998147 | 8 | $c'$ = ,000000003725 | 28 |
| $h$ = ,001905702459 | 9 | $d'$ = ,000000001863 | 29 |
| $i$ = ,000960492403 | 10 | $e'$ = ,000000000931 | 30 |
| $k$ = ,000482856502 | 11 | $f'$ = ,000000000465 | 31 |
| $l$ = ,000242314856 | 12 | $g'$ = ,000000000233 | 32 |
| $m$ = ,000121457237 | 13 | $h'$ = ,000000000116 | 33 |
| $n$ = ,000060829654 | 14 | $i'$ = ,000000000058 | 34 |
| $o$ = ,000030448787 | 15 | $k'$ = ,000000000029 | 35 |
| $p$ = ,000015235790 | 16 | $l'$ = ,000000000015 | 36 |
| $q$ = ,000007621708 | 17 | $m'$ = ,000000000007 | 37 |
| $r$ = ,000003812130 | 18 | $n'$ = ,000000000004 | 38 |
| $s$ = ,000001906491 | 19 | $o'$ = ,000000000002 | 39 |
| $t$ = ,000000953389 | 20 | $p'$ = ,000000000001 | 40 |
| $v$ = ,000000476742 | 21 | | |

# TABLE III.

Sum of $\dfrac{1}{2^n} + \dfrac{1}{4^n} + \dfrac{1}{6^n} +$ &c. *ad infinitum.*

| Sum | $n =$ | Sum | $n =$ |
|---|---|---|---|
| $A'' = ,4112335167 12$ | 2 | $W'' = ,000000238419$ | 22 |
| $B'' = ,1602571\cdot2895$ | 3 | $X'' = ,000000119209$ | 23 |
| $C'' = ,067645202107$ | 4 | $Y'' = ,000000059605$ | 24 |
| $D'' = ,032403992347$ | 5 | $Z'' = ,000000029802$ | 25 |
| $E'' = ,015895985344$ | 6 | $A''' = ,000000014901$ | 26 |
| $F'' = ,007877728730$ | 7 | $B''' = ,000000007450$ | 27 |
| $G'' = ,003922177173$ | 8 | $C''' = ,000000003725$ | 28 |
| $H'' = ,001957047643$ | 9 | $D''' = ,000000001863$ | 29 |
| $I'' = ,000977533765$ | 10 | $E''' = ,000000000931$ | 30 |
| $K'' = ,000488522553$ | 11 | $F''' = ,000000000465$ | 31 |
| $L'' = ,000244200705$ | 12 | $G''' = ,000000000233$ | 32 |
| $M'' = ,000122085292$ | 13 | $H''' = ,000000000116$ | 33 |
| $N'' = ,000061038895$ | 14 | $I''' = ,000000000058$ | 34 |
| $O'' = ,000030518512$ | 15 | $K''' = ,000000000029$ | 35 |
| $P'' = ,000015259024$ | 16 | $L''' = ,000000000015$ | 36 |
| $Q'' = ,000007629452$ | 17 | $M''' = ,000000000007$ | 37 |
| $R'' = ,000003814712$ | 18 | $N''' = ,000000000004$ | 38 |
| $S'' = ,000001907352$ | 19 | $O''' = ,000000000002$ | 39 |
| $T'' = ,000000953675$ | 20 | $P''' = ,000000000001$ | 40 |
| $V'' = ,000000476837$ | 21 | | |

TABLE

# TABLE IV.

Sum of $\dfrac{1}{3^n}+\dfrac{1}{5^n}+\dfrac{1}{7^n}+$ &c. *ad infinitum.*

| Sum | $n =$ | Sum | $n =$ |
|---|---|---|---|
| $a'' = ,233700550136$ | 2 | $n'' = ,000000209240$ | 14 |
| $b'' = ,051799790264$ | 3 | $o'' = ,000000069724$ | 15 |
| $c'' = ,014678031604$ | 4 | $p'' = ,000000023234$ | 16 |
| $d'' = ,004523762760$ | 5 | $q'' = ,000000007744$ | 17 |
| $e'' = ,001447076640$ | 6 | $r'' = ,000000002581$ | 18 |
| $f'' = ,000471548657$ | 7 | $s'' = ,000000000864$ | 19 |
| $g'' = ,000155179025$ | 8 | $t'' = ,000000000286$ | 20 |
| $h'' = ,000051345183$ | 9 | $v'' = ,000000000095$ | 21 |
| $i'' = ,000017041362$ | 10 | $w'' = ,000000000032$ | 22 |
| $k'' = ,000005666051$ | 11 | $x'' = ,000000000011$ | 23 |
| $l'' = ,000001885848$ | 12 | $y'' = ,000000000004$ | 24 |
| $m'' = ,000000628055$ | 13 | $z'' = ,000000000001$ | 25 |

## PROP. I.

*To find the sum of the sums of the reciprocal squares, cubes,*
*&c. &c. ad infinitum.*

By divsion $\dfrac{1}{x-1\times x}=\dfrac{1}{x^2}+\dfrac{1}{x^3}+\dfrac{1}{x^4}+$ &c. *ad inf.*; hence if we make each of these terms the general term of a series, and write 2, 3, 4, &c. *ad inf.* for *x*, we have $\dfrac{1}{1.2}+\dfrac{1}{2.3}+\dfrac{1}{3.4}+$ &c. = (Table 1.) A + B + C + D + &c. ; but $\dfrac{1}{1.2}+\dfrac{1}{2.3}+\dfrac{1}{3.4}+$ &c. *ad inf.*=1 ; hence A + B + C + D + &c. *ad inf.* = 1.

As

As $\dfrac{1}{x \times x + 1} = \dfrac{1}{x^2} - \dfrac{1}{x^3} + \dfrac{1}{x^4} - \dfrac{1}{x^5} + $ &c. *ad inf.* ; we have, by the same method of proceeding, $A - B + C - D + $ &c. *ad inf.* $= \frac{1}{2}$ ; consequently $A + C + E + $ &c. $= \frac{3}{4}$, and $B + D + F + $ &c. $= \frac{1}{4}$.

Because $\dfrac{1}{x - 1 \times x} = \dfrac{1}{x^2} + \dfrac{1}{x^3} + \dfrac{1}{x^4} + $ &c. *ad inf.* ; if for $x$ we write 2, 4, 6, &c., then will $\dfrac{1}{1 \cdot 2} + \dfrac{1}{3 \cdot 4} + \dfrac{1}{5 \cdot 6} + $ &c. $=$ (Tab. 3) $A'' + B'' + C'' + D'' + $ &c. ; but $\dfrac{1}{1 \cdot 2} + \dfrac{1}{3 \cdot 4} + \dfrac{1}{5 \cdot 6} + $ &c. $=$ hyp. log. 2 ; hence $A'' + B'' + C'' + D'' + $ &c. $=$ hyp. log. 2.

If in the same expression we write 3, 5, 7, &c. for $x$, then $\dfrac{1}{2 \cdot 3} + \dfrac{1}{4 \cdot 5} + \dfrac{1}{6 \cdot 7} + $ &c. $=$ (Tab. 4.) $a'' + b'' + c'' + $ &c. ; but $\dfrac{1}{2 \cdot 3} + \dfrac{1}{4 \cdot 5} + \dfrac{1}{6 \cdot 7} + $ &c. $= 1 - $ hyp. log. 2 ; hence $a'' + b'' + c'' + $ &c. $= 1 - $ hyp. log. 2.——Hence from either of these two last cases, we have a very expeditious method of finding the hyp. log. 2.

PROP. II.

*To find the sum of the infinite series whose general term is* $\dfrac{1}{mx^r \pm n}$.

By division $\dfrac{1}{mx^r \pm n} = \dfrac{1}{mx^r} \mp \dfrac{n}{m^2 x^{2r}} + \dfrac{n^2}{m^3 x^{3r}} \mp \dfrac{n^3}{m^4 x^{4r}} + $ &c. *ad inf.* ; hence, if $\dfrac{1}{mx^r \pm n}$ be made the general term of a series, and for $x$ we write 2, 3, 4, &c., its sum will be equal to the sums of another set of serieses, whose terms are the powers of the reciprocals of the natural numbers respectively multiplied

into

into $\frac{1}{m}$, $\frac{n}{m^2}$, $\frac{n^2}{m^3}$, &c.; hence the fum of each of thefe feries being known from the tables, the fum of the given feries will be found.

Ex. 1. Let $\frac{1}{x^2+1}$ be the general term ; now $\frac{1}{x^2+1}=\frac{1}{x^2}-\frac{1}{x^4}+$ $\frac{1}{x^6}-\frac{1}{x^8}+$&c.; hence if for $x$ we write 2, 3, 4, &c. we have $\frac{1}{5}+\frac{1}{10}+\frac{1}{17}+\frac{1}{26}+$&c.$=A-C+E-G+$ &c. = (by Tab. 1.) ,5766740374669.

Ex. 2. Let $\frac{1}{x^2-1}$ be the general term ; then, by the fame method of proceeding, $\frac{1}{3}+\frac{1}{8}+\frac{1}{15}+\frac{1}{24}+$ &c. $=A+C+E+$ &c. = (by Prop. 1.) $\frac{3}{4}$.

Cor. Becaufe $\frac{1}{8}+\frac{1}{24}+\frac{1}{48}+$&c.$=\frac{1}{8}\times\overline{1+\frac{1}{3}+\frac{1}{6}+}$ &c. = (as $1+\frac{1}{3}+\frac{1}{6}+$&c. is the reciprocal of the figurative numbers of the fecond order) $\frac{1}{8}\times2=\frac{1}{4}$; therefore $\frac{1}{3}+\frac{1}{15}+\frac{1}{35}+$ &c. $=\frac{1}{2}$. Alfo, as $\frac{1}{x^2-1}=\frac{1}{x^2}+\frac{1}{x^4}+\frac{1}{x^6}+$&c. ; if we write 2, 4, 6, &c. for $x$, we have $\frac{1}{3}+\frac{1}{15}+\frac{1}{35}+$&c. = (by Tab. 3.) $A''+C''+E''+$ &c. $=\frac{1}{2}$; but, by Prop. 1. $A''+B''+C''+D''+$ &c. = hyp. log. 2; hence $B''+D''+F''+$&c. $=-\frac{1}{2}+$ hyp. log. 2.

Ex. 3. Let the general term be $\frac{1}{x^3-1}=\frac{1}{x^3}+\frac{1}{x^6}+\frac{1}{x^9}+$&c., and, by writing 2, 3, 4, &c. for $x$, we have $\frac{1}{7}+\frac{1}{26}+\frac{1}{63}+$ &c. = $B+E+H+$&c.$=,221689395104.$

<div align="right">Ex.</div>

Ex. 4. Let the general term be $\dfrac{1}{3x^4-2}=\dfrac{1}{3x^4}+\dfrac{2}{9x^5}+\dfrac{4}{27x^{12}}+$ &c.,

and, by writing 2, 3, 4, &c. for $x$, &c. we have $\dfrac{1}{46}+\dfrac{1}{241}+\dfrac{1}{766}+$ &c.

$\dfrac{1}{3}C+\dfrac{2}{9}G+\dfrac{4}{27}L+$ &c. $= ,028385252052.$

Ex. 5. To find the fum of the feries $\dfrac{1}{9}-\dfrac{1}{26}+\dfrac{1}{65}-\dfrac{1}{124}+$ &c.
If we write 2, $-3$, 4, $-5$, &c. for $x$, the general term will
be $\dfrac{1}{x^3+1}=\dfrac{1}{x^3}-\dfrac{1}{x^6}+\dfrac{1}{x^9}-\dfrac{1}{x^{12}}+$ &c. Now, by writing 2, $-3$, 4,

$-5$, &c. for $x$, the feriefes of which $\dfrac{1}{x^3}$, $\dfrac{1}{x^9}$, &c. are the ge-
neral terms, will be alternately $+$ and $-$, and therefore their
fums will be found in Tab. 2. and the feriefes of which

$\dfrac{1}{x^6}$, $\dfrac{1}{x^{12}}$, &c. are the general terms will have their terms all $+$,
and therefore their fums will be found in Tab. 1. Hence
the fum required $= b+h+o+$ &c. $-E-L-R-$ &c. $=$
$,082800931803.$

## PROP. III.

*To find the fum of the fums of the reciprocals of the* odd
*powers in* Tab. 2.

By divifion $\dfrac{1}{x-1\times x}=\dfrac{1}{x^2}+\dfrac{1}{x^3}+\dfrac{1}{x^4}+\dfrac{1}{x^5}+$ &c.; hence by writ-
ing 2, $-3$, 4, $-5$, &c. for $x$, the fums of the feriefes of which
$\dfrac{1}{x^3}$, $\dfrac{1}{x^5}$, &c. are the general terms, may be found by Tab. 2.
and the other fums by Tab. 1.; hence $\dfrac{1}{1.2}+\dfrac{1}{3.4}+\dfrac{1}{5.6}+$ &c.
$= A+C+E+$ &c. $+b+d+f+$ &c.; but $\dfrac{1}{1.2}+\dfrac{1}{3.4}+\dfrac{1}{5.6}+$

&c.

&c. $= -\frac{1}{2} + 2$ hyp. log. 2; and by Prop. 1. $A + C + E + \&c. =$

$\frac{3}{4}$; hence $b + d + f + \&c. = -\frac{5}{4} + 2$ hyp. log. 2.

<div style="text-align:center">PROP. IV.</div>

*To find the sum of the infinite series whose general term is*

$\frac{x^s}{mx^r \pm n}$.

By division $\dfrac{x^s}{mx^r \pm n} = \dfrac{1}{mx^{r-s}} \mp \dfrac{n}{m^2 x^{2r-s}} + \dfrac{n^2}{m^3 x^{3r-s}} \mp \&c.$ *ad inf.*;

hence the sum of the series of which $\dfrac{x^s}{mx^r \pm n}$ is the general term,

is found as in Prop. 2. Here $r$ must be greater than $s$ at least
by 2, otherwise the sum will be infinite.

Ex. 1. Let the general term be $\dfrac{x^2}{x^4 + 1} = \dfrac{1}{x^2} - \dfrac{1}{x^6} + \dfrac{1}{x^{10}} - \&c.$;

hence if for $x$ we write 2, 3, 4, &c. we have $\dfrac{4}{17} + \dfrac{9}{82} + \dfrac{16}{257} +$

$\&c. = A - E + I - N + \&c. = ,538527924723.$—If for $x$ we

write 2, 4, 6, &c. we get $\dfrac{4}{17} + \dfrac{16}{257} + \dfrac{36}{1296} + \&c. = A'' - E'' + I''$

$- N'' + \&c. = ,396257616555.$

Ex. 2. Let the general term be $\dfrac{x}{3x^3 - 1} = \dfrac{1}{3x^2} + \dfrac{1}{9x^5} + \dfrac{1}{27x^8} + \&c.$;

hence if we write 2, 3, 4, &c. for $x$, we have $\dfrac{2}{23} + \dfrac{3}{80} + \dfrac{4}{191} + \&c. =$

$\frac{1}{3}A + \frac{1}{9}D + \frac{1}{27}G + \&c. = ,219238483448.$

By this proposition we may find the sum of any series whose

general term is $\dfrac{ax^s + bx^{s-1} + cx^{s-2} + \&c.}{mx^r \pm n}$; for this resolves itself into

$\dfrac{ax^s}{mx^r \pm n}, \dfrac{bx^{s-1}}{mx^r \pm n},$ &c. &c., the sum of each of which series is

6           found

found by this propofition. Now the $\overline{s+1}$th differences of the numerators of this general term are $=0$, and therefore it comprehends all feries under fuch circumftances. For example, let the given feries be $\frac{4}{17}+\frac{13}{82}+\frac{26}{257}+\frac{43}{626}$. Here the third differences of the numerators $=0$; to find therefore the general expreffion for the numerator, affume $ax^2+bx+c$ for it; and, by writing 2, 3, 4, for $x$, we have $4a+2b+c=4$, $9a+3b+$ $-c=13$, $16a+4b+c=26$; hence $a=2$, $b=-1$, $c=-2$; and as the denominator is manifeftly $x^4+1$, the general term will be $\frac{2x^2-x-2}{x^4+1}=\frac{2x^2}{x^4+1}-\frac{x}{x^4+1}-\frac{2}{x^4+1}$, each of which being made the general term of a feries, their fum will be found to be refpectively 1,077055849446, 0,194173022145, and 0,156955159332; hence the fum of the given feries is 0,725927667969.

If $s$ be negative, the general term becomes $\dfrac{1}{x^s \times mx^r \pm n} = \dfrac{1}{mx^{r+s}}$

$\mp \dfrac{1}{m^2 x^{2r+s}} + \dfrac{1}{m^3 x^{3r+s}} \mp \&c.$

Ex. 1. To find the fum of $\dfrac{1}{1.2.3} - \dfrac{1}{2.3.4} + \dfrac{1}{3.4.5} - \&c.$ *ad inf.* Here the general term is $\dfrac{1}{x-1 \times x \times x+1} = \dfrac{1}{x \times x^2 - 1} =$ $\dfrac{1}{x^3}+\dfrac{1}{x^5}+\dfrac{1}{x^7}+\&c.$; hence, by writing 2, $-3$, 4, $-5$, &c. for $x$, we have the fum $= b+d+f+$ &c. $=$ (by Prop. 3.) $-\dfrac{5}{4}+2$ hyp. log. 2.

If $\dfrac{1}{x-1 \times x^3 \times x+1}$ be the general term it refolves itfelf into $\dfrac{1}{x^5}+\dfrac{1}{x^7}+\dfrac{1}{x^9}+\&c.$; confequently the fum of $\dfrac{1}{1.2^3.3} - \dfrac{1}{2.3^3.4}$

$+$

$+\dfrac{1}{3 \cdot 4^{3} \cdot 5} - \&c. = -b - \dfrac{5}{4} + 2$ hyp. log. 2. In like manner the

fum of $\dfrac{1}{1 \cdot 2^{5} \cdot 3} - \dfrac{1}{2 \cdot 3^{5} \cdot 4} + \dfrac{1}{3 \cdot 4^{5} \cdot 5} - \&c. = -b - d - \dfrac{5}{4} +$

2 hyp. log. 2. Thus we may proceed as far as we pleaſe by adding two powers to the middle term; and hence this remarkable property of the ſerieſes, that the difference of the ſums of the ſerieſes where the middle term is $x$, $x^{3}$, $x^{5}$, &c. is $b$, $d$, $f$, &c. reſpectively.

Ex. 2. In like manner if the general term be $\dfrac{1}{x-1 \times x^{3} \times x+1}$,

and we write 2, 3, 4, &c. for $x$, we have $\dfrac{1}{1 \cdot 2^{3} \cdot 3} + \dfrac{1}{2 \cdot 3^{3} \cdot 3}$

$+ \dfrac{1}{3 \cdot 4^{3} \cdot 5} + \&c. = D + F + H + \&c. =$ (by Prop. 1.) $\dfrac{1}{4} - B$.

Hence alſo $\dfrac{1}{1 \cdot 2^{3} \cdot 3} + \dfrac{1}{2 \cdot 3^{3} \cdot 4} + \&c. = \dfrac{1}{4} - B - D$; and ſo on as before.

If the general term be under the form $\dfrac{1}{x^{n} \cdot x + m}$, it

will be moſt convenient to reſolve it thus: by diviſion

$\dfrac{1}{x+m} = \dfrac{1}{x} - \dfrac{m}{x^{2}} + \dfrac{m^{2}}{x^{3}} - \&c. \pm \dfrac{m^{n}}{x^{n} \cdot x + m}$; hence $\pm \dfrac{1}{x^{n} \cdot x + m} =$

$\overline{\dfrac{1}{x+m} - \dfrac{1}{x} + \dfrac{m}{x^{2}} - \dfrac{m^{2}}{x^{3}} + \&c.} \times \dfrac{1}{m^{n}} = -\dfrac{m}{x \cdot x + m} + \dfrac{m}{x^{2}} - \dfrac{m^{2}}{x^{3}} + \&c. \times \dfrac{1}{m^{n}}$,

where the ſign on the left hand will be $+$ or $-$ according as $n$ is even or odd, and the number of terms on the right is $= n$. Now the ſum of the ſeries whoſe general term is $\dfrac{m}{x \cdot x + m}$ is

well known, and the ſums of the other are found from the tables.

Ex. 1. To find the ſum of $\dfrac{1}{2^{2} \cdot 3} + \dfrac{1}{3^{2} \cdot 4} + \dfrac{1}{4^{2} \cdot 5} + \&c. $ *ad inf.*

Here the general term is $\dfrac{1}{x^2 \times \overline{x+1}} = -\dfrac{1}{x \cdot \overline{x+1}} + \dfrac{1}{x^2}$, and by writing

2, 3. 4, &c. for $x$, we have the sum $= -\dfrac{1}{2 \cdot 3} - \dfrac{1}{3 \cdot 4} - \&c. +$

$A = -\dfrac{1}{2} + A$. In like manner $\dfrac{1}{2^3 \cdot 3} + \dfrac{1}{4^2 \cdot 5} + \dfrac{1}{6 \cdot 7} + \&c. = -1 +$

hyp.log. 2 $+ A''$. Also $\dfrac{1}{2^3 \cdot 5} + \dfrac{1}{3^3 \cdot 6} + \dfrac{1}{4^3 \cdot 7} + \&c. = \overline{\dfrac{13}{12} - 3A + 9B} \times$

$\dfrac{1}{27}$.

If $m$ be negative, then $\dfrac{1}{x^n \cdot \overline{x-m}} = \overline{\dfrac{m}{x \cdot \overline{x-m}} - \dfrac{m}{x^2} - \dfrac{m^2}{x^3} - \&c.} \times \dfrac{1}{m^n}$.

Hence $\dfrac{1}{2^4 \cdot 1} + \dfrac{1}{3^4 \cdot 2} + \dfrac{1}{4^4 \cdot 3} + \&c. = 1 - A - B - C$; and so on for others of the same kind.

If the general term be under this form $\dfrac{1}{x^{rn} \cdot \overline{ax^n + m}}$, then, in

like manner, we have $\pm \dfrac{1}{x^{rn} \cdot \overline{ax^n + m}} = \overline{\dfrac{1}{ax^n + m} - \dfrac{1}{ax^n} + \dfrac{m}{a^2 x^{2n}} - \&c.}$

$\times \dfrac{a^r}{m^r}$, where the sign on the left hand will be $+$ or $-$, according as $r$ is even or odd, and the number of terms on the right is $= r + 1$.

Ex. 1. To find the sum of $\dfrac{1}{2^4 \cdot 5} + \dfrac{1}{3^4 \cdot 10} + \dfrac{1}{4^4 \cdot 17} + \&c.$

Here $m = 1$, $n = 2$, $r = 2$, $a = 1$, and the general term $\dfrac{1}{x^4 \times \overline{x^2 + 1}}$

$= \dfrac{1}{x^2 + 1} - \dfrac{1}{x^2} + \dfrac{1}{x^4}$; now the sum of the series whose general

term is $\dfrac{1}{x^2 + 1}$ is $= ,5766740 37469$, by Prop. 2.; consequently the sum required $= ,5766740 3769 - A + C = ,0 14063204332$.

Ex. 2. If the given series be $\dfrac{1}{4 \cdot 5} + \dfrac{1}{9 \cdot 10} + \dfrac{1}{16 \cdot 17} + \&c.$ the

general

general term will be $\dfrac{1}{x^2 \cdot \overline{x+1}} = -\dfrac{1}{x^2+1} + \dfrac{1}{x^2}$; hence, by writing 2, 3, 4, &c. for $x$, we have the fum $= -{,}576674037469 + A = {,}06826002938.$

If $m$ be negative, then $\dfrac{1}{x^{rn} \cdot \overline{ax^n - m}} = \dfrac{1}{ax^n - m} - \dfrac{1}{ax^n} - \dfrac{m}{a^2 x^{2n}} - \&c,$

$\times \dfrac{a^r}{m^r}.$

Ex. 1. To find the fum of $\dfrac{1}{1 \cdot 2^2 \cdot 3} + \dfrac{1}{2 \cdot 3^2 \cdot 4} + \dfrac{1}{3 \cdot 4^2 \cdot 5} + \&c.$

Here the general term is $\dfrac{1}{x - 1 \times x^2 \times x + 1} = \dfrac{1}{x^2 \times x^2 - 1} = \dfrac{1}{x^2 - 1} - \dfrac{1}{x^2}$;

now, by writing 2, 3, 4, &c. for $x$, the fum of the feries whofe general term is $\dfrac{1}{x^2 - 1}$ is $= \dfrac{3}{4}$, by Prop. 2.; hence the fum required $= \dfrac{3}{4} - A.$

Ex. 2. Let the given feries be $\dfrac{1}{1 \cdot 2^2 \cdot 3} + \dfrac{1}{3 \cdot 4^2 \cdot 5} + \dfrac{1}{5 \cdot 6^2 \cdot 7} +$ &c. Here the general term is the fame as before, writing 2, 4, 6, &c. for $x$; and, by Prop. 2. the fum of the feries whofe general term is $\dfrac{1}{x^2 - 1}$ is $= \dfrac{1}{2}$; hence the fum $= \dfrac{1}{2} - A''.$

Ex. 3. In like manner the fum of the feries $\dfrac{1}{1 \cdot 2^4 \cdot 3} +$ $\dfrac{1}{2 \cdot 3^4 \cdot 4} + \dfrac{1}{3 \cdot 4^4 \cdot 5} + \&c. = {,}221689395104 - B.$

Ex. 4. To find the fum of $\dfrac{1}{3 \cdot 4^2 \cdot 5} + \dfrac{1}{8 \cdot 9^2 \cdot 10} + \dfrac{1}{15 \cdot 16^2 \cdot 17}$ $+ \&c.$ Here the general term is $\dfrac{1}{x^2 - 1 \times x^4 \times x^2 + 1} = \dfrac{1}{x^4 \times x^4 - 1}$

$= \dfrac{1}{x^4 - 1} - \dfrac{1}{x^4}$; but the fum of the feries whofe general term is

$\dfrac{1}{x^4 - 1}$ is $= {,}086662976264$; hence the fum required $=$ ${,}086662976264 - C.$

### P R O P.  V.

*To find the sum of the infinite series* $\frac{1}{15} + \frac{1}{40} + \frac{1}{85} + \frac{1}{156} + \frac{1}{259}$ + &c.

In this series the fourth differences of the denominators $= 0$; therefore the general term must be represented by $\frac{1}{ax^3 + bx^2 + cx + d}$; write therefore 2, 3, 4, &c. for $x$, and we have $8a + 4b + 2c + d = 15$, $27a + 9b + 3c + d = 40$, $64a + 16b + 4c + d = 85$, $125a + 25b + 5c + d = 156$; hence $a = 1$, $b = 1$, $c = 1$, $d = 1$, and the general term is $\frac{1}{x^3 + x^2 + x + 1} = \frac{1}{x^3} - \frac{1}{x^4} + \frac{1}{x^7} - \frac{1}{x^5}$ + &c.; hence the sum $= B - C + F - G + K - L +$ &c. $= ,1242700165$.

### P R O P.  VI.

*To find the sum of* $\frac{2}{5^2} + \frac{3}{10^2} + \frac{4}{17^2}$ + &c. *ad inf.*

The general term $= \frac{x}{x^2 + 1^2} = \frac{1}{x^3} - \frac{2}{x^5} - \frac{3}{x^7}$ + &c.; hence, by writing 2, 3, 4, &c. for $x$, we have the sum $= B - 2D + 3F$ $-$ &c. $= ,147115771469$.

In like manner $\frac{2}{3^2} + \frac{3}{8^2} + \frac{4}{15^2}$ + &c. $= B + 2D + 3F +$ &c. $= ,312498999865$.

### PROP. VII.

*To find the sum of* $\dfrac{1}{3^2 \cdot 5^2} + \dfrac{1}{8^2 \cdot 10^2} + \dfrac{1}{15^2 \cdot 17^2} + \&c.$ *ad inf.*

The general term is $\dfrac{1}{x^2 - 1^2 \times x^2 + 1^2} = \dfrac{1}{x^8} + \dfrac{2}{x^{12}} + \dfrac{3}{x^{16}} + \&c.$ ; hence, by writing 2, 3, 4, &c. for $x$, we have the sum $= G + 2L + 3P + \&c. = ,009447690684.$

### PROP. VIII.

*To find the sum of* $\dfrac{1}{1^3 \cdot 2^3 \cdot 3^3} + \dfrac{1}{2^3 \cdot 3^3 \cdot 4^3} + \dfrac{1}{3^3 \cdot 4^3 \cdot 5^3} + \&c.$ *ad inf.*

Here the general term is $\dfrac{1}{x - 1^3 \cdot x^3 \cdot x + 1^3} = \dfrac{1}{x^9} + \dfrac{3}{x^{11}} + \dfrac{6}{x^{13}} + \&c.$ and hence the sum $= H + 3K + 6M + \&c. = ,004707148337.$

### PROP. IX.

*To find the sum of the infinite series* $1 - \dfrac{1}{3} + \dfrac{1}{6} - \dfrac{1}{10} + \&c.$ *being a series of the reciprocal of the figurative numbers of the $3^{rd}$ order, having the signs alternately $+$ and $-$.*

This series, by resolving two terms into one, becomes $\dfrac{4}{1 \cdot 2 \cdot 3} + \dfrac{4}{3 \cdot 4 \cdot 5} + \dfrac{4}{5 \cdot 6 \cdot 7} + \&c.$ whose general term, by writing 2, 4, 6, &c. for $x$, is $\dfrac{4}{x - 1 \times x \times x + 1} = \dfrac{4}{x^3} + \dfrac{4}{x^5} + \dfrac{4}{x^7} + \&c.$ consequently the sum $= 4B'' + 4D'' + 4F'' + \&c. =$ (by Cor. Ex. 2. Prop. 2.) $- 2 + 4$ hyp. log. $2$.

Cor.

Cor. Hence, as $1 + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \&c. = 2$, we have $1 + \frac{1}{6} +$ $\frac{1}{15} + \&c. = 2$ hyp. log. 2, and $\frac{1}{3} + \frac{1}{10} + \frac{1}{21} + \&c. = 2 - 2$ hyp. log. 2.

## PROP. X.

*To find the sum of the infinite series* $1 - \frac{1}{4} + \frac{1}{10} - \frac{1}{20} + \&c.$ *being the reciprocals of the figurative numbers of the 4th order, having the signs alternately* $+$ *and* $-$ .

If we write $2, -3, 4, -5, \&c.$ for $x$, the general term will be $\frac{6}{x^3 - x} = \frac{6}{x^3} + \frac{6}{x^5} + \frac{6}{x^7} + \&c.$; hence the sum required $=$ $6b + 6d + 6f + \&c. =$ (by Prop. 3.) $- 7\frac{1}{2} + 12$ hyp. log. 2.

Cor. Because the sum of $1 + \frac{1}{4} + \frac{1}{10} + \frac{1}{20} + \&c. = \frac{3}{2}$; therefore $1 + \frac{1}{10} + \frac{1}{35} + \&c. = -3 + 6$ hyp. log. 2; and $\frac{1}{4} + \frac{1}{10} + \frac{1}{56} +$ $\&c. = 4\frac{1}{2} - 6$ hyp. log. 2.

## PROP. XI.

*To find the sum of* $\frac{2^2}{1^2 \cdot 3^2} + \frac{3^2}{2^2 \cdot 4^2} + \frac{4^2}{3^2 \cdot 5^2} + \&c.$ *ad infinitum.*

The general term, by writing $2, 3, 4, \&c.$ for $x$, is $\frac{x^2}{\overline{x-1}^2 \times \overline{x+1}^2} = \frac{1}{x^2} + \frac{2}{x^4} + \frac{3}{x^6} + \&c.$; hence the sum $= A + 2C + 3E$ $+ \&c. = ,8849669934\circ7.$

### PROP. XII.

*To find the sum of* $\dfrac{1}{1 \cdot 2^2 \cdot 3^3} + \dfrac{1}{2 \cdot 3^2 \cdot 4^3} + \dfrac{1}{3 \cdot 4^2 \cdot 5^3} +$ &c. *ad infinitum.*

Here the general term, by writing 2, 3, 4, &c. for $x$, is

$$\frac{1}{x-1 \cdot x^2 \cdot x+1^3} = \frac{1}{x^6} - \frac{2}{x^7} + \frac{4}{x^8} + \frac{6}{x^9} - \frac{9}{x^{10}} - \frac{12}{x^{11}} + \frac{16}{x^{12}} - \text{\&c.} ;$$ conse-

quently the sum $= E - 2F + 4G - 6H + 9I - 12K +$ &c. $=$ ,010370898482.

### PROP. XIII.

*To find the sum of* $\frac{1}{2}A - \frac{1}{4}B + \frac{1}{8}C -$ &c. *ad infinitum.*

The hyp. log. $2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} -$ &c. $= 1 + \frac{1}{3} + \frac{1}{5} +$

&c. $- \frac{1}{2} - \frac{1}{4} - \frac{1}{6} -$ &c. ; hence $2 \times$ hyp. log. 2, or hyp. log. 4,

$= \frac{2}{1} + \frac{2}{3} + \frac{2}{5} +$ &c. $- 1 - \frac{1}{2} - \frac{1}{3} -$ &c. Now, by division,

$\frac{2}{2x+1} = \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{4x^3} - \frac{1}{8x^4} +$ &c. ; hence, by writing 2, 3, 4, &c.

for $x$, we have (after transposition) $\frac{2}{5} + \frac{2}{7} +$ &c. $- \frac{1}{2} - \frac{1}{3} - \frac{1}{4}$

$-$ &c. $= -\frac{1}{2}A + \frac{1}{4}B - \frac{1}{8}C +$ &c. ; hence, by adding equal quantities to each side, we have $\frac{2}{1} + \frac{2}{3} + \frac{2}{5} +$ &c. $- \frac{1}{2} - \frac{1}{3} - \frac{1}{4} -$

&c. $= \frac{8}{3} - \frac{1}{2}A + \frac{1}{4}B - \frac{1}{8}C +$ &c., consequently $\frac{1}{2}A - \frac{1}{4}B + \frac{1}{8}C -$

&c. $= \frac{8}{3} - \frac{2}{1} - \frac{2}{3} - \frac{2}{5} -$ &c. $+ 1 + \frac{1}{2} + \frac{1}{3} +$ &c. $= \frac{8}{3} -$ hyp. log. 4.

PROP. XIV.

*To find the sum of the infinite series* $\frac{1}{2 \cdot 5} + \frac{1}{3 \cdot 7} + \frac{1}{4 \cdot 9} + $ &c.

The general term, by writing 2, 3, 4, &c. for $x$, is $\frac{1}{x \cdot 2x+1} = \frac{1}{2x^2} - \frac{1}{4x^3} + \frac{1}{8x^4} - $ &c.; hence the sum $= \frac{1}{2}A - \frac{1}{4}B + \frac{1}{8}C$ $-$ &c. $=$ (by Prop. 13.) $\frac{8}{3} - $ hyp. log. 4.

PROP. XV.

*To find the sum of* $1 + \frac{1}{2} + \frac{1}{3} + \cdots \cdots to \frac{1}{x}$.

The hyp. log. $\frac{x}{x-1} = \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{3x^3} + \frac{1}{4x^4} + $ &c.; confequently hyp. log. $\frac{x}{x-1} - \frac{1}{2x^2} - \frac{1}{3x^3} - \frac{1}{4x^4} - $ &c. $= \frac{1}{x}$; hence, if we write 2, 3, 4, &c. for $x$, we have hyp. log. $\frac{2}{1} + $ hyp. log. $\frac{3}{2} + $ &c. . . . .

$$\left. \begin{array}{l} \text{hyp. log.} \; \frac{x}{x-1} - \overline{\frac{1}{2} \times \frac{1}{2^2} + \frac{1}{3^2} + \&c. \ldots \ldots \frac{1}{x^2}} \\ \qquad - \frac{1}{3} \times \frac{1}{2^3} + \frac{1}{3^3} + \&c. \ldots \ldots \frac{1}{x^3} \\ \qquad - \frac{1}{4} \times \frac{1}{2^4} + \frac{1}{3^4} + \&c. \ldots \ldots \frac{1}{x^4} \\ \qquad - \quad \&c. \quad \&c. \qquad \&c. \end{array} \right\} =$$

$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + $ &c. . . . . $\frac{1}{x}$; but hyp. log. $\frac{2}{1} + $ hyp. log. $\frac{3}{2} + $ hyp. log. $\frac{4}{3} + $ &c. . . . . hyp. log. $\frac{x}{x-1} = $ hyp. log. $\frac{2}{1} \times \frac{3}{2} \times \frac{4}{3} \times $ &c. . . . . $\frac{x}{x-1} = $ hyp. log. $x$; alfo $\frac{1}{2^2} + \frac{1}{3^2} + $ &c. . . . . $\frac{1}{x^2} = $ the

7                                                    fum

fum of the fame feries *ad infinitum, minus* the fum of all the terms

from $\frac{1}{x^2} = $ (if $x + 1 = n$) $A - \frac{1}{n} - \frac{1}{2n^2} - \frac{1}{6n^3} + \frac{1}{30n^5} - \frac{1}{42n^7} + $ &c.; in

the fame manner $\frac{1}{2^3} + \frac{1}{3^3} + $ &c.... $\frac{1}{x^3} = B - \frac{1}{2n^2} - \frac{1}{2n^3} - \frac{1}{4n^4} + \frac{1}{12n^6}$

$[- \frac{1}{12n^8} + $ &c.; and fo on for the other feriefes; hence, by fub-

ftitution, and adding unity to each fide, we have hyp. log.

$x + 1 - \frac{1}{2}A - \frac{1}{3}B - \frac{1}{4}C - $ &c. $+ \frac{1}{2n} + \frac{5}{12n^2} + \frac{1}{3n^3} + \frac{31}{120n^4} + \frac{1}{5n^5} + \frac{41}{252n^6}$

$+ \frac{1}{7n^7} + \frac{31}{240n^8} + $ &c. $= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + $ &c.... $\frac{1}{x}$; but $1 - \frac{1}{2}A -$

$\frac{1}{3}B - \frac{1}{4}C - $ &c. $= ,5772156649901$; hence $1 + \frac{1}{2} + \frac{1}{3} + $ &c.

... $\frac{1}{x} = $ hyp. log. $x + ,5772156649901 + \frac{1}{2n} + \frac{5}{12n^2} + \frac{1}{3n^3} + \frac{31}{120n^4}$

$+ \frac{1}{5n^5} + \frac{41}{252n^6} + \frac{1}{7n^7} + \frac{31}{240n^8} + $ &c.

Ex. 1. Let $x = 10000$; then

$$\text{hyp. log. } 10000 = 9,2103403719976$$
$$\text{conft. quant. } = ,5772156664901$$
$$\frac{1}{2n} = ,000049995000$$
$$\frac{5}{12n^2} = ,000000004166$$

therefore the fum required $= 9,7876060036043$

Ex. 2. Let $x = 10000000$; then

$$\text{hyp. log. } 10000000 = 16,1180956650958$$
$$\text{conft. quant. } = ,5772156664901$$
$$\frac{1}{2n} = ,000000049999$$

therefore the fum required $= 16,6953113365858$

PROP.

## PROP. XVI.

*To find the value of* $\alpha \times \beta \times \gamma \times \delta \times$ &c. *ad infinitum, supposing the general term to be a rational function of* x.

Let $\pi$ be the general term, then resolve $\frac{\dot{\pi}}{\pi}$ into an infinite series, and take the fluent on both sides; then write 2, 3, 4, &c. for $x$, and one side will become the hyp. log. of the given series, and the value of the other side may be found from the tables.

Ex. 1. To find the value of $\frac{4}{3} \times \frac{9}{8} \times \frac{16}{15} \times$ &c. *ad infinitum.*

Here the general term is $\frac{x^2}{x^2-1}$; hence $\frac{\dot{\pi}}{\pi} = -\frac{2\dot{x}}{x^3-x} = -\frac{2\dot{x}}{x^3}$

$-\frac{2\dot{x}}{x^5} - \frac{2\dot{x}}{x^7} -$ &c.; hence the hyp. log. $\frac{x^2}{x^2-1} = \frac{1}{x^2} + \frac{1}{2x^4} + \frac{1}{3x^6} +$ &c.

Write 2, 3, 4, &c. for $x$, and we have the hyp. log. $\frac{4}{3} +$ hyp. log.

$\frac{9}{8} +$ hyp. log. $\frac{16}{15} +$ &c. $= A + \frac{1}{2}C + \frac{1}{3}E +$ &c. $= ,693147180574,$

which is the hyp. log. 2; but hyp. log. $\frac{4}{3} +$ hyp. log. $\frac{9}{8} +$

hyp. log. $\frac{16}{15} +$ &c. $=$ hyp. log. $\frac{4}{3} \times \frac{9}{8} \times \frac{16}{15} \times$ &c. consequently

$\frac{4}{3} \times \frac{9}{8} \times \frac{16}{15} \times$ &c. $= 2.$

Ex. 2. To find the value of $\frac{8}{7} \times \frac{27}{26} \times \frac{64}{63} \times$ &c. *ad infinitum.*

Here the general term is $\frac{x^3}{x^3-1}$; hence $\frac{\dot{\pi}}{\pi} = -\frac{3\dot{x}}{x^4-x} = -\frac{3\dot{x}}{x^4} -$

$\frac{3\dot{x}}{x^7} - \frac{3\dot{x}}{x^{10}} -$ &c.; hence the hyp. log. $\frac{x^3}{x^3-1} = \frac{1}{x^3} + \frac{1}{2x^6} + \frac{1}{3x^9} +$ &c.

Write 2, 3, 4, &c. for $x$, and we have hyp. log. $\frac{8}{7} +$ hyp. log.

$\frac{27}{26}$

$\frac{27}{26}$+hyp. log. $\frac{64}{63}$+&c. = B+$\frac{1}{2}$E+$\frac{1}{3}$H+&c. = ,211466250444;

or hyp. log. $\frac{8}{7}$ × $\frac{27}{26}$ × $\frac{64}{63}$ × &c. = ,211466250444; hence $\frac{8}{7}$ ×]

$\frac{27}{26}$ × $\frac{64}{63}$ × &c. = 1,627295, &c.

Hence we may find the value of such a quantity, suppofing the number of factors to be finite.

Ex. To find the value of $\frac{2}{1}$ × $\frac{4}{3}$ × $\frac{6}{5}$ × &c. .... to $\frac{2x}{2x-1}$.

Here the general term being $\frac{2x}{2x-1}$, we have $\frac{\dot{\pi}}{\pi} = -\frac{\dot{x}}{2x^2 - x} = -$

$\frac{\dot{x}}{2x^2} - \frac{\dot{x}}{4x^3} - \frac{\dot{x}}{8x^4} - \frac{\dot{x}}{16x^5}$ − &c.; hence hyp. log. $\frac{2x}{2x-1} = \frac{1}{1 \cdot 2x} +$

$\frac{1}{2 \cdot 4x^2} + \frac{1}{3 \cdot 8x^3} + \frac{1}{4 \cdot 16x^4}$ + &c. Now write 2, 3, 4, &c.

for $x$, and we have the hyp. log. $\frac{4}{3}$ × $\frac{6}{5}$ × $\frac{8}{7}$ × &c. ....

$$\left.\begin{array}{l}\dfrac{2x}{2x-1} = \dfrac{1}{2} \times \overline{\dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \cdots \cdot \dfrac{1}{x}} \\[2mm] + \dfrac{1}{2 \cdot 4} \times \overline{\dfrac{1}{2^2} + \dfrac{1}{3^2} + \dfrac{1}{4^2} + \cdots \cdot \dfrac{1}{x^2}} \\[2mm] + \dfrac{1}{3 \cdot 8} \times \overline{\dfrac{1}{2^3} + \dfrac{1}{3^3} + \dfrac{1}{4^3} + \cdots \cdot \dfrac{1}{x^3}} \\[2mm] + \quad \&c. \qquad \&c. \qquad\quad \&c.\end{array}\right\}$$

But, by Prop. 15. $\frac{1}{2}$ × $\overline{\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \cdot \frac{1}{x}} = \frac{1}{2}$hyp. log.

$x - ,211392167549 + \frac{1}{4n} + \frac{5}{24n^2} + \frac{1}{6n^3} + \frac{31}{240n^4} + \frac{1}{10n^5} + \frac{41}{504n^6}$ + &c. ;

alfo $\frac{1}{2 \cdot 4}$ × $\overline{\frac{1}{2^2} + \frac{1}{3^2} + \cdots \cdot \frac{1}{x^2}} = \frac{1}{2 \cdot 4}$ × $\overline{A - \frac{1}{n} - \frac{1}{2n^2} - \frac{1}{6n^3} + \frac{1}{30n^5}}$ −

&c. and $\frac{1}{3 \cdot 8}$ × $\overline{\frac{1}{2^3} + \frac{1}{3^3} + \cdots \cdot \frac{1}{x^3}} = \frac{1}{3 \cdot 8}$ × $\overline{B - \frac{1}{2n^2} - \frac{1}{2n^3} - \frac{1}{4n^4}} +$

$\overline{\frac{1}{12n^6}}$ − &c., and fo on for the other feriefes : hence, by fubftitu-

tion,

tion, hyp. log. $\dfrac{4}{3} \times \dfrac{6}{5} \times \dfrac{8}{7} \times$ &c. . . . . $\dfrac{2x}{2x-1} = \frac{1}{2}$ hyp. log. $x -$

,12078223764 $+ \dfrac{1}{8n} + \dfrac{1}{8n^2} + \dfrac{23}{192n^3} + \dfrac{7}{64n^4} + \dfrac{61}{640n^5}$ $+$ &c.; confe-

quently $\dfrac{4}{3} \times \dfrac{6}{5} \times \dfrac{8}{7} \times$ &c. . . . . $\dfrac{2x}{2x-1} =$ the natural number

corresponding to the right hand side of the equation; hence

$\dfrac{2}{1} \times \dfrac{4}{3} \times \dfrac{6}{5} \times$ . . . . $\dfrac{2x}{2x-1} =$ twice that natural number.

    Ex. Let $x = 10000$; then

        $\frac{1}{2}$ hyp. log. $x = 4,605170185988$

        const. quant. $=$ ,120782237640

          ———————————————

            4,484387948348

       $\dfrac{1}{8n} =$ ,000012498750

       $\dfrac{1}{8n^2} =$ ,000000001249

         ———————————————

            4,484400448347 the natural number

corresponding to which hyp. log. is 88,6238, &c., confe-

quently $\dfrac{2}{1} \times \dfrac{4}{3} \times \dfrac{6}{5} \times$ . . . . . $\dfrac{20000}{19999} = 177,2476.$

    If $x$ be a very large number, it may be sufficiently exact in most cafes to take twice the natural number corresponding to the hyp. log. of $\frac{1}{2}$ hyp. log. $x -$ ,120782237640.